

**PARTIALLY LOCALIZED QUASIMODES IN LARGE  
SUBSPACES  
(PRELIMINARY VERSION – COMMENTS WELCOME)**

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*Abstract:* We consider spaces of high-energy quasimodes for the Laplacian on a compact hyperbolic surface, and show that when the spaces are large enough, one can find quasimodes that exhibit strong localization phenomena. Namely, take any constant  $c$ , and a sequence of  $cr_j$ -dimensional spaces  $\mathcal{S}_j$  of quasimodes, where  $\frac{1}{4} + r_j^2 \rightarrow \infty$  is an approximate eigenvalue for  $\mathcal{S}_j$ . Then we can find a sequence of vectors  $\psi_j \in \mathcal{S}_j$ , such that any weak-\* limit point of the microlocal lifts of  $|\psi_j|^2$  localizes a positive proportion of its mass on a singular set of codimension 1. This result is sharp, in light of the QUE result of [BL11] for certain joint quasimodes that include spaces of size  $o(r_j)$ , with arbitrarily slow decay.

## 1. INTRODUCTION

The Quantum Unique Ergodicity (QUE) Conjecture of Rudnick-Sarnak [RS94] states that eigenfunctions of the Laplacian on Riemannian manifolds of negative sectional curvature become equidistributed in the high-energy limit. Although there exist so-called “toy models” of quantum chaos that do not exhibit this behavior (see eg. [FNDB03, AN07, Kel07]), it has been suggested that large degeneracies of the quantum propagator may be responsible for some of these phenomena (see eg. [Sar11]). Since the Laplacian on a surface of negative curvature is not expected to have large degeneracies, one can explore this aspect and introduce “degeneracies” by considering quasimodes, or approximate eigenfunctions, in place of true eigenfunctions—relaxing the order of approximation to true eigenfunctions yields larger spaces of quasimodes, mimicking higher-dimensional eigenspaces. Studying the properties of such quasimodes— and, especially, the effect on equidistribution of varying the order of approximation— can help shed light on the overall role of spectral degeneracies in the theory.

Let  $X = \Gamma \backslash SL(2, \mathbb{R}) = S^*M$  compact. We normalize the uniform measure  $dz$  on  $M$  (and the measure  $dx$  on  $X$ ) to have total volume 1.

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We define an  $\omega(r)$ -**quasimode with approximate parameter**  $r$  to be a function  $\psi$  satisfying

$$\|(\Delta + (\frac{1}{4} + r^2))\psi\|_2 \leq r\omega(r)\|\psi\|_2$$

The factor of  $r$  in our definition comes from the fact that  $r$  is essentially the square-root of the Laplace eigenvalue. For any constant  $C$ , denote by  $S_C(r)$  the space spanned by eigenfunctions of spectral parameter in  $[r - C, r + C]$ ; then for large  $r$  (in particular,  $r > C$ ), any vector in  $S_C(r)$  is a  $3C$ -quasimode of approximate parameter  $r$ .

For a sequence of spectral parameters  $r_j \rightarrow \infty$ , consider the space  $S_C(r_j) \subset C^\infty(\Gamma \backslash \mathbb{H})$ , spanned by the eigenfunctions whose spectral parameters lie in  $[r_j - C, r_j + C]$ ; coarse estimates on the remainder in Weyl's Law show that this space has dimension  $\dim S_C(r_j) \gtrsim Cr_j$ . To any vector  $\psi \in S_C(r_j)$  we associate a measure  $|\Psi|^2 dVol$  on  $S^*M$  (see section 2.2) called the **microlocal lift** of  $|\psi|^2 dz$ . A sequence  $\{\psi_j \in S_C(r_j)\}$  is said to satisfy the QUE property if the measures  $|\Psi_j|^2 dVol$  converge weak-\* to the uniform (Liouville) measure on  $S^*M$ .

In joint work with Lindenstrauss [BL11], we studied certain cases of *joint quasimodes*—eg., of the Laplacian and one Hecke operator—and found that QUE held for all sequences of functions, that were jointly  $o(1)$ -quasimodes for both the Laplacian and one Hecke operator. It is important to note that there are spaces of such joint  $o(1)$  quasimodes of size  $o(r)$  with arbitrarily slow decay; these are considerably larger than the spaces of Laplace-quasimodes that are expected to satisfy QUE without any Hecke assumption. This is a testament to the rigidity imposed by the additional structure of the Hecke correspondence, as already apparent in [Lin06].

Here we show that, in fact, these  $o(r)$ -dimensional spaces are as large as possible for QUE.

**Theorem 1.** *Let  $\mathcal{S}_j \subset S_C(r_j)$  be a subspace of dimension  $cr_j$  for each  $S_C(r_j)$ . Then there exists a sequence of quasimodes  $\psi_j \in \mathcal{S}_j$ , such that any weak-\* limit point of the microlocal lifts  $|\Psi_j|^2 dVol$  gives positive measure to a subset of  $X$  of codimension 1.*

The results of [BL11] show that for these special arithmetic manifolds, there can be subspaces of  $S_C(r_j)$ , of dimension  $o(r_j)$ , such that any sequence of quasimodes taken from these spaces must satisfy QUE. Theorem 1 shows that if the subspaces of quasimodes are taken to be any larger—i.e., of dimension  $\geq cr_j$  for some fixed constant  $c$ —then one can always find bad sequences that do not satisfy QUE. In this sense, the special joint  $o(1)$ -quasimodes of [BL11] are optimally degenerate for QUE.

The codimension 1 subset of  $X = \Gamma \backslash SL(2, \mathbb{R})$  we have in mind is the collection of geodesic segments through a single base point  $p \in \Gamma \backslash \mathbb{H}$ ; i.e., cotangent vectors pointing radially towards or away from  $p$ , at distance  $\leq \tau$ , where  $\tau$  is a fixed number depending only on the manifold  $M$ . For each  $j$ , we will find a base point  $p_j \in \Gamma \backslash \mathbb{H}$  and a quasimode  $\psi_j \in \mathcal{S}_j$  that is large at  $p_j$ , and thus the microlocal lift  $|\Psi_j|^2 dVol$  (see section 2) will be enhanced on vectors pointing radially relative to  $p_j$ . We will then take  $p$  to be a limit point of  $\{p_j\}$  in the compact manifold  $\Gamma \backslash \mathbb{H}$ .

The idea is to use a kernel that is spectrally localized near  $S_C(r_j)$ , and spatially localized near radial vectors around  $p_j$ ; we describe such a kernel in section 3. Since the kernel is spectrally localized near  $S_C(r_j)$ , it will strongly correlate with our  $\Psi_j$  (up to factors depending on  $c$ ,  $C$ , and the manifold  $M$ ), and show that the latter must also localize a fixed positive proportion of its mass near our codimension 1 subset.

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## 2. MICROLOCAL LIFTS OF QUASIMODES

**2.1. Some Harmonic Analysis on  $SL(2, \mathbb{R})$ .** We begin by reviewing some harmonic analysis on  $SL(2, \mathbb{R})$  that we will need. Throughout, we write  $X = \Gamma \backslash SL(2, \mathbb{R})$  and  $M = \Gamma \backslash \mathbb{H} = \Gamma \backslash SL(2, \mathbb{R})/K$ , where  $K = SO(2)$  is the maximal compact subgroup.

Fix an orthonormal basis  $\{\phi_l\}$  of  $L^2(M)$  consisting of Laplace eigenfunctions, which we can take to be real-valued for simplicity. Each eigenfunction generates, under right translations, an irreducible representation  $V_l = \overline{\{\phi_l(xg^{-1}) : g \in SL(2, \mathbb{R})\}}$  of  $SL(2, \mathbb{R})$ , which span a dense subspace of  $L^2(X)$ .

We distinguish the pairwise orthogonal **weight spaces**  $A_{2n}$  in each representation, consisting of those functions satisfying  $f(xk_\theta) = e^{2in\theta} f(x)$  for all  $k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in K$  and  $x \in X$ . The weight spaces together span a dense subspace of  $V_l$ . Each weight space is one-dimensional in  $V_l$ , spanned by  $\phi_{2n}^{(l)}$  where

$$\begin{aligned} \phi_0^{(l)} &= \phi_l \in A_0 \\ (ir_l + \frac{1}{2} + n)\phi_{2n+2}^{(l)} &= E^+ \phi_{2n}^{(l)} \\ (ir_l + \frac{1}{2} - n)\phi_{2n-2}^{(l)} &= E^- \phi_{2n}^{(l)} \end{aligned}$$

Here  $E^+$  and  $E^-$  are the **raising** and **lowering operators**, first-order differential operators corresponding to  $\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})$  and  $\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})$  in the Lie algebra. The normalized pseudodifferential operators

$$\begin{aligned} R &= \frac{E^+}{ir_l + \frac{1}{2} + n} : \phi_{2n}^{(l)} \mapsto \phi_{2n+2}^{(l)} \\ R^{-1} &= \frac{E^-}{ir_l + \frac{1}{2} - n} : \phi_{2n}^{(l)} \mapsto \phi_{2n-2}^{(l)} \end{aligned}$$

are unitary and left-invariant, and each  $\phi_{2n}$  is a unit vector. We define the distribution

$$\Phi_\infty^{(l)} = \sum_{n=-\infty}^{\infty} \phi_{2n}^{(l)}$$

and extend this definition by linearity to  $\Psi_\infty = \sum_{n=-\infty}^{\infty} \psi_{2n} = \sum_{n=-\infty}^{\infty} R^n \psi$  for  $\psi \in S_C(r)$ , where each  $\psi_{2n} = R^n \psi$ .

Let  $k$  be a  $K$ -bi-invariant function on  $SL(2, \mathbb{R})$ — i.e., a radial function on  $\mathbb{H}$ . Then any Laplace eigenfunction  $\phi$  of eigenvalue  $\frac{1}{4} + s^2$  is also an eigenfunction of convolution with  $k$ , with eigenvalue given by the spherical transform  $h(s)$  of  $k$ . The spherical transform is related to  $k$  by the Selberg/Harish-Chandra transform (see eg. [Iwa02, Chapter 1.8])

$$\begin{aligned} h(s) &= \int_{-\infty}^{\infty} e^{isu} g(u) du \\ g(u) &= 2Q\left(\sinh^2\left(\frac{u}{2}\right)\right) \\ (1) \quad k(t) &= -\frac{1}{\pi} \int_t^\infty \frac{dQ(\omega)}{\sqrt{\omega - t}} \end{aligned}$$

The coordinate  $t(z, w) = 2 \sinh^2(d(z, w)/2)$  is often more convenient for calculations ( $dt$  is the radial volume measure on  $\mathbb{H}$ ). What is most important for our purposes is that whenever  $g$  is compactly supported in the interval  $[-\tau, \tau]$ , the kernel  $k$  will be supported in the ball of radius  $\tau$  in  $\mathbb{H}$ .

We can write such a  $k$  as a (left- $K$ -invariant) function on  $\mathbb{H}$ , and use Helgason's Fourier inversion [Hel81] to write

$$k(z) = \int_{s=0}^{\infty} \int_B e^{(is + \frac{1}{2}) \langle z, b \rangle} \hat{k}(s, b) s \tanh(\pi s) ds db$$

where  $b \in B$  runs over the boundary  $S^1$  of the disc model for  $\mathbb{H}$ , and  $\langle z, b \rangle$  represents the (signed) distance to the origin from the horocycle through the point  $z \in \mathbb{H}$  tangent to  $b \in B$ . Since each plane wave  $e^{(-is+\frac{1}{2})\langle \cdot, b \rangle}$  is an eigenfunction of spectral parameter  $s$ , the Fourier transform

$$\begin{aligned}\hat{k}(s, b) &= \int_{\mathbb{H}} e^{(-is+\frac{1}{2})\langle z, b \rangle} k(z) dz \\ &= h(s)\end{aligned}$$

so that

$$k(z) = \int_{s=0}^{\infty} \left( \int_B e^{(is+\frac{1}{2})\langle z, b \rangle} db \right) h(s) s \tanh(\pi s) ds$$

It will be more convenient to write as in [Zel87]

$$e^{(is+\frac{1}{2})\langle z, b \rangle} db = e^{(is-\frac{1}{2})\langle z, b \rangle} d\theta = e^{(is-\frac{1}{2})\varphi(g.k_\theta)} d\theta$$

where  $\varphi(g)$  is the signed distance from the origin to the horocycle through  $g \in SL(2, \mathbb{R})$ , and  $k_\theta$  parametrizes the  $SO(2)$  fibre  $gK$ . Since  $\varphi$  is left  $K$ -invariant and right  $N$ -invariant, it is convenient to use  $KAN$  coordinates to write

$$g = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that this distance is given by  $\varphi(g) = t = \log(a^2 + c^2)$ .

**2.2. Construction of the Microlocal Lifts.** We set

$$I_\psi(f) = \langle Op(f)\psi, \psi \rangle := \langle f\Psi_\infty, \psi \rangle = \lim_{N \rightarrow \infty} \left\langle f \sum_{n=-N}^N \psi_{2n}, \psi_0 \right\rangle$$

according to the pseudo-differential calculus of [Zel87], which clearly restricts to the measure  $|\psi(z)|^2 dz$  when applied to  $K$ -invariant functions  $f \in C^\infty(M)$ , by orthogonality of the weight spaces. Note moreover that this limit is purely formal for  $K$ -finite  $f$ , and since these  $K$ -finite functions are dense in the space of smooth functions, we can restrict our attention to these. We denote by  $A_{2n}$  the  $n$ -th weight space, consisting of smooth functions that transform via  $f(xk_\theta) = e^{2in\theta} f(x)$  for all  $x \in X$ .

**Lemma 1.** *Let  $\psi \in S_C(r)$  be a unit vector, and set*

$$\Psi := \sqrt{\frac{3L}{2L^2 + 1}} \sum_{|n| \leq \sqrt{r}} \frac{L - |n|}{L} \psi_{2n}$$

for  $L := \lfloor \sqrt{r} \rfloor$ . Then for any  $K$ -finite  $f \in \sum_{n=-N_0}^{N_0} A_{2n}$ , we have

$$I_\psi(f) = \langle f\Psi, \Psi \rangle + O_{f,C}(r^{-1/2})$$

Note that the prefactor  $\sqrt{\frac{3L}{2L^2+1}} \sim \sqrt{\frac{3}{2}}r^{-1/4}$  is simply an  $L^2$ -normalization of the Fejér coefficients  $\frac{L-|n|}{L}$ . The proof of the Lemma is identical to that of [BL11], the only difference being this use of Fejér coefficients in place of the Dirichlet coefficients  $\frac{1}{\sqrt{2L+1}}$  used in [Lin01] and [BL11]. The extra smoothness provided by using Fejér coefficients will be exploited in section 3.

*Proof:* First, we wish to show that

$$\langle f\psi_{2n}, \psi_{2m} \rangle = \langle f\psi_{2n+2}, \psi_{2m+2} \rangle + O_{f,C}(r^{-1})$$

for all  $-\sqrt{r} \leq n, m \leq \sqrt{r}$ , satisfying  $|n - m| \leq N_0$  (if the latter condition is not met, both inner products are trivial, by orthogonality of the weight spaces). We will work individually with each pair of spectral components of  $\psi$ , and then re-average over the spectral decomposition; therefore, we write  $\psi^{(r_1)}$  and  $\psi^{(r_2)}$  for the projections of  $\psi$  to the eigenspaces of parameters  $r_1$  and  $r_2$ , respectively. Recall that  $r_1, r_2 = r + O_C(1)$  by the condition  $\psi \in S_C(r_j)$ .

We have

$$\begin{aligned} & \langle f\psi_{2n}^{(r_1)}, \psi_{2m}^{(r_2)} \rangle \\ &= \frac{1}{(ir_1 - n - \frac{1}{2})(-ir_2 - m - \frac{1}{2})} \langle fE^- \psi_{2n+2}^{(r_1)}, E^- \psi_{2m+2}^{(r_2)} \rangle \\ &= \frac{1}{(ir_1 - n - \frac{1}{2})(-ir_2 - m - \frac{1}{2})} \left( \langle E^-(f\psi_{2n+2}^{(r_1)}), E^- \psi_{2m+2}^{(r_2)} \rangle - \langle E^-(f)\psi_{2n+2}^{(r_1)}, E^- \psi_{2m+2}^{(r_2)} \rangle \right) \\ &= -\frac{1}{(ir_1 - n - \frac{1}{2})(-ir_2 - m - \frac{1}{2})} \langle f\psi_{2n+2}^{(r_1)}, E^+ E^- \psi_{2m+2}^{(r_2)} \rangle + O(r_1^{-1}) \langle E^-(f)\psi_{2n+2}^{(r_1)}, \psi_{2m}^{(r_2)} \rangle \\ &= -\frac{(-ir_2 + m + \frac{1}{2})(-ir_2 - m - \frac{1}{2})}{(ir_1 - n - \frac{1}{2})(-ir_2 - m - \frac{1}{2})} \langle f\psi_{2n+2}^{(r_1)}, \psi_{2m+2}^{(r_2)} \rangle + O(r_1^{-1}) \langle E^-(f)\psi_{2n+2}^{(r_1)}, \psi_{2m}^{(r_2)} \rangle \\ &= \langle f\psi_{2n+2}^{(r_1)}, \psi_{2m+2}^{(r_2)} \rangle + \frac{i(r_2 - r_1) + (n - m)}{ir_1 - n - \frac{1}{2}} \langle f\psi_{2n+2}^{(r_1)}, \psi_{2m+2}^{(r_2)} \rangle + O(r_1^{-1}) \langle E^-(f)\psi_{2n+2}^{(r_1)}, \psi_{2m}^{(r_2)} \rangle \end{aligned}$$

We now average over  $r_2$ , and since  $i(r_2 - r_1) + (n - m) = O_{f,C}(1)$  (recall that  $|n - m| \leq N_0 = O_f(1)$ ), we have

$$\begin{aligned} & \langle f\psi_{2n}^{(r_1)}, \psi_{2m} \rangle - \langle f\psi_{2n+2}^{(r_1)}, \psi_{2m+2} \rangle \\ &= \frac{1}{ir_1 - n - \frac{1}{2}} \left\langle f\psi_{2n+2}^{(r_1)}, \sum_{r_2} O_{f,C}(1) \psi_{2m+2}^{(r_2)} \right\rangle + O(r_1^{-1}) \langle E^-(f)\psi_{2n+2}^{(r_1)}, \psi_{2m} \rangle \end{aligned}$$

and further averaging over  $r_1$  gives

$$\begin{aligned} & \langle f\psi_{2n}, \psi_{2m} \rangle - \langle f\psi_{2n+2}, \psi_{2m+2} \rangle \\ &= \left\langle f \sum_{r_1} O(r_1^{-1}) \psi_{2n+2}^{(r_1)}, \sum_{r_2} O_{f,C}(1) \psi_{2m+2}^{(r_2)} \right\rangle + \left\langle E^-(f) \sum_{r_1} O(r_1^{-1}) \psi_{2n+2}^{(r_1)}, \psi_{2m} \right\rangle \end{aligned}$$

Now the error terms are estimated by Cauchy-Schwarz, giving

$$\begin{aligned} & \left\langle f \sum_{r_1} O(r_1^{-1}) \psi_{2n+2}^{(r_1)}, \sum_{r_2} O_{f,C}(1) \psi_{2m+2}^{(r_2)} \right\rangle \\ & \leq \|f\|_\infty \left\| \sum_{r_1} O(r_1^{-1}) \psi_{2n+2}^{(r_1)} \right\|_2 \left\| \sum_{r_2} O_{f,C}(1) \psi_{2m+2}^{(r_2)} \right\|_2 \\ & \leq O_{f,C}(r_1^{-1}) \|\psi_{2n+2}\|_2 \|\psi_{2m+2}\|_2 \\ & = O_{f,C}(r_1^{-1}) \end{aligned}$$

and

$$\begin{aligned} \langle E^-(f) \sum_{r_1} O(r_1^{-1}) \psi_{2n+2}^{(r_1)}, \psi_{2m} \rangle & \leq \|E^-(f)\|_\infty O(r_1^{-1}) \|\psi_{2n+2}\|_2 \|\psi_{2m}\|_2 \\ & = O_f(r_1^{-1}) \end{aligned}$$

by using the orthogonality of the  $\psi_{2n+2}^{(r_1)}$  to estimate

$$\begin{aligned} \left\| \sum_{r_1} O(r_1^{-1}) \psi_{2n+2}^{(r_1)} \right\|_2^2 &= O(r_1^{-2}) \sum_{r_1} \|\psi_{2n+2}^{(r_1)}\|_2^2 \\ &= O(r_1^{-2}) \|\psi_{2n+2}\|_2^2 \end{aligned}$$

and similarly

$$\left\| \sum_{r_2} O_{f,C}(1) \psi_{2m+2}^{(r_2)} \right\|_2^2 = O_{f,C}(1) \|\psi_{2m+2}\|_2^2$$

Therefore

$$\langle f\psi_{2n}, \psi_{2m} \rangle = \langle f\psi_{2n+2}, \psi_{2m+2} \rangle + O_{f,C}(r^{-1})$$

since  $|r_1 - r| \leq C$  implies that  $O_{f,C}(r_1^{-1}) = O_{f,C}(r^{-1})$ .

We iterate this  $|m| \leq \sqrt{r}$  times, arriving at

$$(2) \quad \langle f\psi_{2n}, \psi_{2m} \rangle = \langle f\psi_{2(n-m)}, \psi_0 \rangle + O_{f,C}(\sqrt{r}r^{-1})$$

Now, by definition

$$\begin{aligned}\langle f\Psi, \Psi \rangle &= \sum_{|m|, |n| \leq L} \frac{3(L - |n|)(L - |m|)}{2L^3 + L} \langle f\psi_{2n}, \psi_{2m} \rangle \\ &= \sum_{n=-L}^L \sum_{m=n-N_0}^{n+N_0} \frac{3(L - |n|)(L - |m|)}{2L^3 + L} \langle f\psi_{2n}, \psi_{2m} \rangle - \sum_{|n|=L-N_0}^L O_f(N_0 L^{-1})\end{aligned}$$

since each term  $\frac{3(L - |n|)(L - |m|)}{2L^3 + L} \langle f\psi_{2n}, \psi_{2m} \rangle = O_f(L^{-1})$ , and for each value of  $n$ , there are at most  $N_0$  values of  $m$  such that the inner product is not trivial. Thus, since  $N_0 = O_f(1)$ , we get

$$\begin{aligned}\langle f\Psi, \Psi \rangle &= \sum_{n=-L}^L \sum_{m=n-N_0}^{n+N_0} \frac{3(L - |n|)(L - |m|)}{2L^3 + L} \langle f\psi_{2n}, \psi_{2m} \rangle + O_f(L^{-1}) \\ &= \frac{3}{2L^3 + L} \sum_{n=-L}^L \sum_{m=n-N_0}^{n+N_0} \left( (L - |n|)^2 - O(L|n - m|) \right) \langle f\psi_{2n}, \psi_{2m} \rangle + O_f(L^{-1}) \\ &= \left( \frac{3}{2L^3 + L} \sum_{n=-L}^L (L - |n|)^2 \right) \sum_{n-m=-N_0}^{N_0} \left( \langle f\psi_{2(n-m)}, \psi_0 \rangle + O_{f,C}(r^{-1/2}) \right) + O_f(L^{-1})\end{aligned}$$

by (2). Therefore setting  $l = n - m$  we finally obtain

$$\begin{aligned}\langle f\Psi, \Psi \rangle &= \sum_{l=-N_0}^{N_0} \langle f\psi_{2l}, \psi_0 \rangle + O_{f,C}(r^{-1/2}) + O_f(L^{-1}) \\ &= I_{\psi_j} + O_{f,C}(r^{-1/2})\end{aligned}$$

since  $L \leq r^{-1/2}$ .  $\square$

For any given sequence  $\{\psi_j \in S_C(r_j)\}$ , we have constructed a sequence  $\{\Psi_j\}_{j=1}^\infty$  such that the microlocal lifts  $|\Psi_j|^2 dVol$  are asymptotically equivalent to the distributions  $I_{\psi_j}$ . It is these measures that we wish to study.

### 3. A “MICROLOCAL” KERNEL

Pick once and for all an orthonormal basis  $\{\phi_l\}$  of  $L^2(M)$  consisting of real-valued eigenfunctions. Throughout, we will allow all implied constants to depend on  $C$ ,  $c$ , and the manifold  $M$ .

We will need an auxilliary spherical kernel  $k$ . Observe that the hypotheses of Theorem 1 are weaker when  $C$  is larger and  $c$  smaller; in particular, we may assume that  $C$  is sufficiently large. We begin with

$$\tilde{h}(s) = r_j^{-1/2} \frac{\cosh s/C \cosh r_j/C}{\cosh 2s/C + \cosh 2r_j/C}$$



a scaled version of the spherical transform used in [IS95]. The Fourier transform of  $h$  is

$$\tilde{g}(\xi) = \frac{C}{4} r_j^{-1/2} \frac{\cos(\xi r_j)}{\cosh\left(\frac{C\pi\xi}{2}\right)}$$

It will be convenient to cutoff  $\tilde{g}$ , so that our kernel will have compact support. Let  $\tau$  be a small— but fixed— number less than the radius of injectivity of  $M$ , and pick a smooth, non-negative, even cutoff function  $\chi \in C_c^\infty([-\tau, \tau])$  whose Fourier transform  $\hat{\chi}$  is also non-negative; we normalize so that  $\hat{\chi}(s) \geq 1$  for all  $|s| \leq 1$ . Note that  $\chi$  depends only on  $\tau$ , and is independent of the parameters  $C$  and  $c$ , as well as  $r_j$ . Now let

$$g(\xi) = \tilde{g}(\xi)\chi(\xi) \in C_c^\infty([-\tau, \tau])$$

whereby the corresponding kernel  $k$  given by (1) will be compactly supported inside the ball of radius  $\tau$  in  $\mathbb{H}$ .

The spherical transform  $h = \tilde{h} * \hat{\chi}$  satisfies

$$\begin{aligned} \|h\|_\infty &\leq \|\hat{\chi}\|_{L^1} \cdot \|\tilde{h}\|_\infty \\ (3) \quad &\lesssim_\tau r_j^{-1/2} \end{aligned}$$

Moreover, since  $\hat{\chi} \geq 1$  on  $[-1, 1]$ , we have for  $|s - r_j| \leq C$

$$\begin{aligned} \min_{|s-r_j| \leq C} h(s) &\geq \inf_{|s'-s| \leq 1} \tilde{h}(s) \geq \inf_{|s'-r_j| \leq (C+1)} r_j^{-1/2} \frac{\cosh \frac{s'}{C} \cosh \frac{r_j}{C}}{\cosh \frac{2s'}{C} + \cosh \frac{2r_j}{C}} \\ &\geq \inf_{|s'-r_j| \leq (C+1)} r_j^{-1/2} \frac{1}{4} \frac{e^{s'/C + r_j/C}}{2 \max\{e^{2s'/C}, e^{2r_j/C}\}} \\ &\geq \inf_{|s'-r_j| \leq (C+1)} \frac{1}{8} r_j^{-1/2} e^{-|s'-r_j|/C} \\ (4) \quad &\geq \frac{1}{100} r_j^{-1/2} \end{aligned}$$

provided  $C > 1$  (which we may assume). Thus  $h$  is large on  $[r_j - C, r_j + C]$ , and our kernel will correlate well with quasimodes in  $S_C(r_j)$ .

On the other hand, since  $\tilde{h}$  decays rapidly away from  $\pm r_j$ , and  $\chi \in C^\infty$  implies that  $\hat{\chi}$  decays rapidly, the convolution  $h$  decays away from  $\pm r_j$  as well; to be precise, the estimate  $h(s) \lesssim r_j^{-1/2} |s - r_j|^{-3}$  will suffice for our purposes. Since  $h$  is even, we can take  $s \geq 0$  without

loss of generality, and estimate first for  $s > r_j$

$$\begin{aligned}
h(s) &= \int_{-\infty}^{\infty} \tilde{h}(r) \hat{\chi}(s-r) dr \\
&\leq \|\tilde{h}\|_{\infty} \int_{-\infty}^{\frac{s+r_j}{2}} \hat{\chi}(s-r) dr + \|\hat{\chi}\|_{\infty} \int_{\frac{s+r_j}{2}}^{\infty} \tilde{h}(r) dr \\
&\leq r_j^{-1/2} \int_{u=\frac{s-r_j}{2}}^{\infty} \hat{\chi}(u) du + \|\hat{\chi}\|_{\infty} \int_{u=\frac{s-r_j}{2}}^{\infty} \tilde{h}(r_j+u) du \\
&\lesssim r_j^{-1/2} |s-r_j|^{-3} + r_j^{-1/2} \exp(-|s-r_j|/2C) \\
&\lesssim r_j^{-1/2} |s-r_j|^{-3}
\end{aligned}$$

since  $\hat{\chi}(u) \lesssim |u|^{-4}$  for some uniform constant. Similarly if  $0 < s < r_j$  we have

$$\begin{aligned}
h(s) &\leq \|\tilde{h}\|_{\infty} \int_{-\infty}^{-r_j/2} \hat{\chi}(s-r) dr + \|\hat{\chi}\|_{\infty} \int_{-r_j/2}^{\frac{r_j+s}{2}} \tilde{h}(r) dr + \|\tilde{h}\|_{\infty} \int_{\frac{r_j+s}{2}}^{\infty} \hat{\chi}(s-r) dr \\
&\lesssim r_j^{-1/2} |s-r_j|^{-3}
\end{aligned}$$

Therefore, we can estimate various spectral integrals that will be needed in the argument: we clearly have

$$(5) \quad \int_0^{\infty} h(s) ds \lesssim r_j^{-1/2}$$

and moreover

$$\begin{aligned}
\int_0^{\infty} sh(s) ds &\lesssim r_j^{-1/2} \left( \int_0^{r_j-1} \frac{s}{(r_j-s)^3} ds + \int_{r_j-1}^{r_j+1} s ds + \int_{r_j+1}^{\infty} \frac{s}{(s-r_j)^3} ds \right) \\
&\lesssim r_j^{-1/2} \left( \int_{u=1}^{r_j} \frac{r_j-u}{u^3} du + r_j + \int_{u=1}^{\infty} \frac{r_j+u}{u^3} du \right) \\
(6) \quad &\lesssim r_j^{1/2}
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_0^{\infty} sh(s)^2 ds &\lesssim r_j^{-1} \left( \int_0^{r_j-1} \frac{s}{(r_j-s)^6} ds + \int_{r_j-1}^{r_j+1} s ds + \int_{r_j+1}^{\infty} \frac{s}{(s-r_j)^6} ds \right) \\
(7) \quad &\lesssim 1
\end{aligned}$$

We set  $k$  to be the radial kernel corresponding to  $h$ , which by (1) is supported in the ball of radius  $\tau$  in  $\mathbb{H}$ . We can estimate  $\|k\|_{L^2(\mathbb{H})}$  by

unitarity of the Helgason Fourier transform, giving

$$\begin{aligned} \|k\|_{L^2}^2 &= \int_0^\infty h(s)^2 s \tanh(\pi s) ds \\ &\leq \int_0^\infty h(s)^2 s ds \lesssim 1 \end{aligned}$$

by (7). Since  $k$  is compactly supported inside the ball of radius  $\tau$ , which is less than the radius of injectivity of  $M$ , we can periodicize to obtain

$$k_p(z) = \sum_{\gamma \in \Gamma} k(p^{-1}\gamma z)$$

as a function on  $M$ , and if  $\mathcal{F}_\Gamma \subset \mathbb{H}$  is a fundamental domain for  $M$ , we have  $k_p(z) = k(p^{-1}z)$  whenever  $p, z \in \mathcal{F}_\Gamma$ .

We wish to show that the microlocal lift  $|\kappa_p|^2 dVol$  of  $|k_p|^2 dz$  is concentrated on cotangent vectors pointing radially towards and away from the base point  $p$ . Precisely, let  $\mathcal{B}_{r_j} \subset S^*\mathbb{H} = KAN$  be given by

$$\mathcal{B}_{r_j} = \{KAN_u : |u| \leq Nc^{-1}r_j^{-1/2}\} \cap B(0, \tau)K$$

for a constant  $N$  to be chosen later; here  $B(0, \tau)K = S^*\{d(z, 0) < \tau\} \subset S^*\mathbb{H}$ , where  $\tau$  was chosen above to be less than the radius of injectivity of  $M$ . Thus,  $\mathcal{B}_{r_j}$  is an  $O(r_j^{-1/2})$ -neighborhood of the union of geodesic segments through 0 up to distance  $\tau$ , with the implied constant depending on  $N$  and  $c$ . Define à la Lemma 1

$$\kappa := b(r_j)r_j^{-1/4} \sum_{|n| \leq L = \lfloor \sqrt{r_j} \rfloor} \frac{L - |n|}{L} R^n k$$

where  $R : \phi_{2n}^{(r)} \mapsto \phi_{2n+2}^{(r)}$  and  $R^{-1} : \phi_{2n}^{(r)} \mapsto \phi_{2n-2}^{(r)}$  are the unitary raising and lowering operators as in section 2. The prefactor  $b(r_j) \sim \sqrt{3/2}$  normalizes  $\|\kappa\|_{L^2(S^*\mathbb{H})} = \|k\|_{L^2(\mathbb{H})}$ . Note also that, since the summation over  $\Gamma$  acting on the left of  $g$  commutes with the left-invariant operators  $R^n$ , we have

$$\begin{aligned} \kappa_p(g) &= b(r_j)r_j^{-1/4} \sum_{|n| \leq L = \lfloor \sqrt{r_j} \rfloor} \frac{L - |n|}{L} R^n k_p \\ &= \sum_{\gamma \in \Gamma} \kappa(p^{-1}\gamma \cdot g) \end{aligned}$$

Additionally, since the unitary, left-invariant operators  $R^n$  descend to  $M$ , and the weight spaces are orthogonal, we have

$$\|\kappa_p\|_{L^2(S^*M)} = \|k_p\|_{L^2(M)} = \|k\|_{L^2(\mathbb{H})} \lesssim 1$$

Since the distribution

$$\sum_{n=-\infty}^{\infty} R^n k(g) = \int_{s=0}^{\infty} e^{(is-\frac{1}{2})\varphi(g.k_\theta)} d\theta h(s) s \tanh(\pi s) ds$$

by [Zel87], the function  $\kappa \in C^\infty(SL(2, \mathbb{R}))$  is given by

$$\kappa(g) = b(r_j) \int_{s=0}^{\infty} \frac{1}{2\pi} \int_{\theta=0}^{2\pi} e^{(is-\frac{1}{2})\varphi(g.k_\theta)} r_j^{-1/4} F_{\lfloor \sqrt{r_j} \rfloor}(\theta) d\theta h(s) s \tanh(\pi s) ds$$

where<sup>1</sup>  $F_L = \frac{1}{L} \left( \frac{\sin L\theta/2}{\sin \theta/2} \right)^2$  is the Fejér kernel of order  $L$ , and  $\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \log(a^2 + c^2)$  as in section 2.1.

The next two Lemmas establish the key property, that  $\kappa_p$  mainly lives in  $p\mathcal{B}_{r_j}$ .

**Lemma 2.** *Let  $g = k_\alpha \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in B(0, \tau)K \setminus \mathcal{B}_{r_j}$ ; in particular,  $|t| \leq \tau$  and  $Nc^{-1}r_j^{-1/2} \leq |n| \lesssim_\tau 1$ . Then*

$$\frac{d}{d\theta} \varphi(g.k_\theta) \gtrsim n$$

for all  $|\theta| \leq |n|N^{-1/4}$ .

*Proof:* Since  $\varphi$  is left  $K$ -invariant, it is sufficient to prove this for  $g = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{t/2} & e^{t/2}n \\ 0 & e^{-t/2} \end{pmatrix}$ . Write

$$\begin{aligned} g.k_\theta &= \begin{pmatrix} e^{t/2} & e^{t/2}n \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} e^{t/2} \cos \theta + e^{t/2}n \sin \theta & -e^{t/2} \sin \theta + e^{t/2}n \cos \theta \\ e^{-t/2} \sin \theta & e^{-t/2} \cos \theta \end{pmatrix} \end{aligned}$$

so that we wish to evaluate

$$\begin{aligned} \frac{d}{d\theta} \varphi(g.k_\theta) &= \frac{d}{d\theta} \log((e^{t/2} \cos \theta + e^{t/2}n \sin \theta)^2 + (e^{-t/2} \sin \theta)^2) \\ &= \frac{d}{d\theta} \log(e^t \cos^2 \theta + ne^t \sin 2\theta + e^t n^2 \sin^2 \theta + e^{-t} \sin^2 \theta) \\ &= \frac{(-4 \sinh t + e^t n^2) \sin 2\theta + 2ne^t \cos 2\theta}{e^t \cos^2 \theta + O_{n,t}(\theta)} \end{aligned}$$

---

<sup>1</sup>The standard normalization of the Fejér kernel is  $\|F_L\|_{L^1} = 1$ , whereas our vectors are  $L^2$ -normalized, which causes a number of  $r^{1/4}$  factors to appear throughout the discussion.

Since  $|\theta| \leq |n|N^{-1/4}$ , and  $n$  and  $t$  are uniformly bounded (depending only on  $\tau$ ), we have  $|(-4 \sinh t + e^t n^2) \sin 2\theta| \lesssim_\tau |n|N^{-1/4}$ . Since moreover the denominator is bounded below by  $\frac{1}{2}e^{-\tau}$ , the Lemma holds as soon as  $N$  is large enough relative to  $\tau$ .  $\square$

**Lemma 3.** *With notations as above, we can choose  $N$  sufficiently large (independent of  $r_j$ ) so that  $\mathcal{B}_{r_j}$  satisfies*

$$\int_{X \setminus p\mathcal{B}_{r_j}} |\kappa_p(y)|^2 dy \leq \frac{1}{50,000} c$$

*Remark:* Recall that

$$\|\kappa_p\|_{L^2(S^*M)}^2 = \|k\|_{L^2(\mathbb{H})}^2 \asymp 1$$

So Lemma 3 is essentially saying that by letting  $N$  be sufficiently large (independent of  $r_j$ ), we can get a large percentage of the  $L^2$ -mass of  $\kappa_p$  to lie inside  $p\mathcal{B}_{r_j}$ .

*Proof:* The main step is bounding  $\int |\kappa|^2$  on  $B(0, \tau)K \setminus \mathcal{B}_{r_j}$ . For this, we will decompose  $\kappa$  into two pieces— the contribution of low frequencies, and that of high frequencies— and estimate each individually on  $B(0, \tau)K \setminus \mathcal{B}_{r_j}$ .

Write  $\kappa = \kappa_1 + \kappa_2$ , where

$$\begin{aligned} \kappa_1(g) &= b(r_j) \int_{s=0}^{\eta r_j} \frac{1}{2\pi} \int_{\theta=0}^{2\pi} e^{(is - \frac{1}{2})\varphi(g.k_\theta)} r_j^{-1/4} F_{[\sqrt{r_j}]}(\theta) d\theta h(s) s \tanh(\pi s) ds \\ \kappa_2(g) &= b(r_j) \int_{s=\eta r_j}^{\infty} \frac{1}{2\pi} \int_{\theta=0}^{2\pi} e^{(is - \frac{1}{2})\varphi(g.k_\theta)} r_j^{-1/4} F_{[\sqrt{r_j}]}(\theta) d\theta h(s) s \tanh(\pi s) ds \end{aligned}$$

where  $\eta$  is chosen small enough, depending on  $C$  and  $c$ , so that computing as in (7)

$$\begin{aligned} \int_{s=0}^{\eta r_j} h(s)^2 s \tanh(\pi s) ds &\leq r_j^{-1} \int_{s=0}^{\eta r_j} \frac{s}{(r_j - s)^6} ds \\ &\leq r_j^{-1} \int_{(1-\eta)r_j}^{r_j} \frac{r_j - u}{u^6} du \\ &< \frac{1}{200,000} c \end{aligned}$$

Thus

$$\begin{aligned} \int_{B(0, \tau)K \setminus \mathcal{B}_{r_j}} |\kappa_1|^2 &\leq \|\kappa_1\|_{L^2(S^*\mathbb{H})}^2 \leq \|k_1\|_{L^2(\mathbb{H})}^2 \\ (8) \qquad \qquad \qquad &\leq \int_{s=0}^{\eta r_j} h(s)^2 s \tanh(\pi s) ds < \frac{1}{200,000} c \end{aligned}$$

for  $k_1(z) = \int_{s=0}^{\eta r_j} \frac{1}{2\pi} \int_{\theta=0}^{2\pi} e^{(is-\frac{1}{2})\varphi(z.k_\theta)} d\theta h(s) s \tanh(\pi s) ds$ , by unitarity of the Helgason Fourier transform.

We turn to  $\kappa_2$  on  $B(0, \tau)K \setminus \mathcal{B}_{r_j}$ ; we will estimate  $|\kappa_2|$  pointwise. For any  $s > \eta r_j$ , and any  $g \in KA \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  with  $|n| \geq Nc^{-1}r_j^{-1/2}$ , write

$$\begin{aligned} & \left| \int_{\theta=0}^{2\pi} e^{(is-\frac{1}{2})\varphi(g.k_\theta)} F_{\lfloor \sqrt{r_j} \rfloor}(\theta) d\theta \right| \\ & \leq \int_{|\theta| \geq |n|/N^{1/4}} |F_{\lfloor \sqrt{r_j} \rfloor}(\theta)| e^{-\frac{1}{2}\varphi(g.k_\theta)} d\theta + \sum_{|m| \leq \sqrt{r_j}} \left| \int_{|\theta| \leq |n|/N^{1/4}} e^{i(s\varphi(g.k_\theta)+m\theta)} e^{-\frac{1}{2}\varphi(g.k_\theta)} d\theta \right| \end{aligned} \quad (9)$$

Now since  $e^{-\frac{1}{2}\varphi(g.k_\theta)}$  and its derivatives are uniformly bounded on  $B(0, \tau)K$ , we may apply a non-stationary phase argument to determine that each integral in the sum on the right is  $O(s^{-1}|n|^{-1})$  whenever the derivative  $s\varphi'(g.k_\theta) + m$  of the phase function is  $\gtrsim s|n|$ : setting  $a(\theta) = e^{-\frac{1}{2}\varphi(g.k_\theta)}$ , and recalling that  $\varphi$  and  $a$ , and their derivatives, are uniformly bounded on  $B(0, \tau)K$ , we write

$$\begin{aligned} & \left| \int_{|\theta| \leq |n|/N^{1/4}} e^{i(s\varphi(g.k_\theta)+m\theta)} a(\theta) d\theta \right| \\ & = \left| \int_{|\theta| \leq |n|/N^{1/4}} \frac{1}{i(s\varphi'(g.k_\theta) + m)} \frac{d}{d\theta} [e^{i(s\varphi(g.k_\theta)+m\theta)}] a(\theta) d\theta \right| \\ & = \left| \int_{|\theta| \leq |n|/N^{1/4}} \frac{d}{d\theta} [e^{i(s\varphi(g.k_\theta)+m\theta)}] \frac{a}{i(s\varphi'(g.k_\theta) + m)} d\theta \right| \\ & \leq 2 \sup \left| \frac{a}{s\varphi'(g.k_\theta) + m} \right| + \int_{|\theta| \leq |n|/N^{1/4}} \left| \frac{a'(s\varphi' + m) - as\varphi''}{(s\varphi' + m)^2} \right| d\theta \\ & \lesssim s^{-1}|n|^{-1} + \int_{|\theta| \leq |n|/N^{1/4}} \frac{s}{(s\varphi' + m)^2} d\theta \\ & \lesssim s^{-1}|n|^{-1} + s^{-1}|n|^{-2} \int_{|\theta| \leq |n|/N^{1/4}} d\theta \\ & \lesssim s^{-1}|n|^{-1} \end{aligned}$$

But Lemma 2 shows that  $s \frac{d\varphi(g.k_\theta)}{d\theta} \gtrsim s|n| > \eta Nc^{-1}r_j^{1/2} > 2|m|$  for all  $|\theta| \leq |n|/N^{1/4}$  if  $N$  is chosen large enough, so that indeed  $s\varphi' + m \gtrsim$

$s|n|$ . Therefore

$$\begin{aligned} \sum_{|m| \leq \sqrt{r_j}} \left| \int_{|\theta| \leq |n|/N^{1/4}} e^{i(s\varphi(g.k_\theta) + m\theta)} e^{-\frac{1}{2}\varphi(g.k_\theta)} d\theta \right| &\lesssim \sum_{|m| \leq \sqrt{r_j}} s^{-1}|n|^{-1} \\ &\lesssim r_j^{1/2} s^{-1} |n|^{-1} \end{aligned}$$

To estimate the  $|F_{\lfloor \sqrt{r_j} \rfloor}|$  term of (9), observe that since  $\varphi$  is bounded on  $B(0, \tau)K$

$$\begin{aligned} \int_{|\theta| \geq |n|/N^{1/4}} |F_{\lfloor \sqrt{r_j} \rfloor}(\theta)| e^{-\frac{1}{2}\varphi(g.k_\theta)} d\theta &\lesssim \int_{|\theta| \geq |n|/N^{1/4}} |F_{\lfloor \sqrt{r_j} \rfloor}(\theta)| d\theta \\ &\lesssim \int_{|\theta| \geq |n|/N^{1/4}} \frac{1}{\sqrt{r_j}} \left( \frac{\sin(\lfloor \sqrt{r_j} \rfloor \theta/2)}{\sin \frac{\theta}{2}} \right)^2 d\theta \\ &\lesssim \int_{|\theta| \geq |n|/N^{1/4}} r_j^{-1/2} \theta^{-2} d\theta \lesssim N^{1/4} r_j^{-1/2} |n|^{-1} \end{aligned}$$

so that combining the two parts of (9) we have

$$(10) \quad \left| \int_{\theta=0}^{2\pi} e^{(is - \frac{1}{2})\varphi(g.k_\theta)} F_{\lfloor \sqrt{r_j} \rfloor}(\theta) d\theta \right| \leq N^{1/4} |n|^{-1} (r_j^{-1/2} + r_j^{1/2} s^{-1})$$

Therefore, since

$$\begin{aligned} |\kappa_2(g)| &\leq \int_{s=\eta r_j}^{\infty} \frac{1}{2\pi} \left| \int_{\theta=0}^{2\pi} e^{(is - \frac{1}{2})\varphi(g.k_\theta)} F_{\lfloor \sqrt{r_j} \rfloor}(\theta) d\theta \right| r_j^{-1/4} h(s) s \tanh(\pi s) ds \\ &\lesssim N^{1/4} |n|^{-1} r_j^{-1/2} \int_{s=\eta r_j}^{\infty} (r_j^{-1/4} s h(s) + r_j^{3/4} h(s)) ds \\ &\lesssim N^{1/4} r_j^{-1/4} |n|^{-1} \end{aligned}$$

by (5) and (6), we see that

$$\begin{aligned} \int_{B(0, \tau)K \setminus \mathcal{B}_{r_j}} |\kappa_2|^2 &\leq \int_{|n| \geq N c^{-1} r_j^{-1/2}} |\kappa_2|^2 \\ &\lesssim N^{1/2} r_j^{-1/2} \int_{|n| \geq N c^{-1} r_j^{-1/2}} |n|^{-2} \\ &\lesssim N^{1/2} r_j^{-1/2} \cdot (N^{-1} c r_j^{1/2}) \\ &\lesssim N^{-1/2} c \end{aligned}$$

and we now choose  $N$  large enough (depending on the implied constant, which in turn depends on the parameters  $\tau$ ,  $C$ , and  $c$ ) so that

$$(11) \quad \int_{B(0, \tau)K \setminus \mathcal{B}_{r_j}} |\kappa_2|^2 < \frac{1}{200,000} c$$

Combining (8) and (11) we get

$$\begin{aligned}
\int_{S^*M \setminus p\mathcal{B}_{r_j}} |\kappa(p^{-1}g)|^2 dg &= \int_{B(0,\tau)K \setminus \mathcal{B}_{r_j}} |\kappa(g)|^2 dg \\
&\leq 2 \int_{B(0,\tau)K \setminus \mathcal{B}_{r_j}} (|\kappa_1|^2 + |\kappa_2|^2) dg \\
&\leq \frac{1}{50,000}^c
\end{aligned}$$

as required.  $\square$

#### 4. PROOF OF THEOREM 1

Equipped with this construction, we can return to the main result.

*Proof of Theorem 1:* The next order of business is to find a quasimode  $\psi_j \in \mathcal{S}_j$ , and a point  $p_j \in \Gamma \setminus \mathbb{H}$  at which  $|\psi_j|$  is large. The following observation can be found in [Sar], in the context of constructing an eigenfunction of large  $L^\infty$ -norm in a highly degenerate eigenspace.

Consider

$$\int_M \sum_{\phi_l \in \mathcal{S}_j} |\phi_l(z)|^2 dx = \sum_{\phi_l \in \mathcal{S}_j} \int_M |\phi_l(z)|^2 dx = \sum_{\phi_l \in \mathcal{S}_j} 1 = cr_j$$

This implies that there exists a point  $p_j \in M$ , such that

$$\sum_{\phi_l \in \mathcal{S}_j} |\phi_l(p_j)|^2 \geq cr_j$$

is at least as large as the average value. Therefore the quasimode

$$\psi_j := \sum_{\phi_l \in \mathcal{S}_j} \phi_l(p_j) \phi_l$$

satisfies

$$(12) \quad \psi_j(p_j) = \|\psi_j\|_2^2 = \sum_{\phi_l \in \mathcal{S}_j} |\phi_l(p_j)|^2 \geq \sqrt{cr_j} \|\psi_j\|_2$$

This is the sequence  $\psi_j$  of quasimodes we will use.

Now consider  $\mathcal{B}_j := p_j \mathcal{B}_{r_j}$ , and since

$$\|\kappa_{p_j}\|_{L^2(\mathcal{B}_j)}^2 \leq \|\kappa_{p_j}\|_{L^2(S^*M)}^2 = \|k_{p_j}\|_{L^2(M)}^2 \lesssim 1$$



we have by Cauchy-Schwarz

$$\begin{aligned}
\int_{\mathcal{B}_j} |\Psi_j(x)|^2 dx &\geq \frac{1}{\|\kappa_{p_j}\|_{L^2(\mathcal{B}_j)}^2} \left( \int_{\mathcal{B}_j} |\Psi_j(x)| |\kappa_{p_j}(x)| dx \right)^2 \\
&\gtrsim \left( \langle \Psi_j, \kappa_{p_j} \rangle - \|\kappa_{p_j}\|_{L^2(X \setminus \mathcal{B}_j)} \|\Psi_j\|_2 \right)^2 \\
&\gtrsim \left( \langle \Psi_j, \kappa_{p_j} \rangle - \frac{1}{200} \sqrt{c} \|\Psi_j\|_2 \right)^2
\end{aligned}$$

since  $\int_{X \setminus \mathcal{B}_j} |\kappa_{p_j}|^2 \leq \frac{1}{50,000} c$  by Lemma 3.

Now since the normalized raising and lowering operators  $R^n$  are unitary, and the weight spaces orthogonal, we have  $\langle \Psi_j, \kappa_{p_j} \rangle = \langle \psi_j, k_{p_j} \rangle$ , so that writing the spectral expansion  $k_{p_j}(z) = \sum_{\phi_l} h(s_l) \phi_l(p_j) \phi_l(z)$ , we use (4) to get

$$\begin{aligned}
\langle \Psi_j, \kappa_{p_j} \rangle &= \langle \psi_j, k_{p_j} \rangle \\
&= \sum_{\phi_l \in \mathcal{S}_j} h(s_l) |\phi_l(p_j)|^2 \\
&\geq \frac{1}{100} r_j^{-1/2} \sum_{\phi_l \in \mathcal{S}_j} |\phi_l(p_j)|^2 \\
&\geq \frac{1}{100} r_j^{-1/2} \cdot (c r_j)^{1/2} \|\Psi_j\|_2 = \frac{1}{100} \sqrt{c} \|\Psi_j\|_2
\end{aligned}$$

Therefore

$$\int_{\mathcal{B}_j} |\Psi_j(x)|^2 dx \gtrsim \left( \left[ \frac{1}{100} - \frac{1}{200} \right] \sqrt{c} \|\Psi_j\|_2 \right)^2 \gtrsim \|\Psi_j\|_2^2$$

with implied constant depending on  $c$ ,  $C$ , and  $\tau$ . In other words, the  $L^2$ -mass of  $\Psi_j$  inside  $\mathcal{B}_j$  is at least a fixed percentage of the total mass, independent of  $r_j \rightarrow \infty$ .

To finish, suppose we have a subsequence of  $\{\Psi_j\}$  such that  $d\mu_j = |\Psi_j|^2 dVol$  converges weak-\* to a measure  $\mu$ , and pick a further subsequence of the  $j$ 's so that  $p_j \rightarrow \bar{p} \in \Gamma \setminus \mathbb{H}$ . Consider any fixed neighborhood  $U \subset X$  of the (compact) union of geodesic segments of length  $\tau$  through  $\bar{p}$ ; it is evident that  $U$  can be chosen to have arbitrarily small volume in  $X$ . On the other hand, any such neighborhood must contain  $\mathcal{B}_j$  for sufficiently large  $j$  in our subsequence, whereby  $\mu_j(U) = \int_U |\Psi_j(x)|^2 dx \gtrsim \|\Psi_j\|_2^2$  for all sufficiently large  $r_j$ . Thus the measure  $\mu$  concentrates a positive proportion of its mass on this codimension 1 subset.  $\square$

## REFERENCES

- [AN07] Nalini Anantharaman and Stéphane Nonnenmacher. Entropy of semiclassical measures of the Walsh-quantized baker's map. *Ann. Henri Poincaré*, 8(1):37–74, 2007.
- [BL11] Shimon Brooks and Elon Lindenstrauss. Joint quasimodes, positive entropy, and quantum unique ergodicity, preprint, 2011.
- [FNDB03] Frédéric Faure, Stéphane Nonnenmacher, and Stephan De Bièvre. Scarred eigenstates for quantum cat maps of minimal periods. *Comm. Math. Phys.*, 239(3):449–492, 2003.
- [Hel81] Sigurdur Helgason. *Topics in harmonic analysis on homogeneous spaces*, volume 13 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1981.
- [IS95] H. Iwaniec and P. Sarnak.  $L^\infty$  norms of eigenfunctions of arithmetic surfaces. *The Annals of Mathematics*, 141(2):301–320, 1995.
- [Iwa02] Henryk Iwaniec. *Spectral methods of automorphic forms*, volume 53 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2002.
- [Kel07] Dubi Kelmer. Scarring on invariant manifolds for perturbed quantized hyperbolic toral automorphisms. *Comm. Math. Phys.*, 276(2):381–395, 2007.
- [Lin01] Elon Lindenstrauss. On quantum unique ergodicity for  $\Gamma \backslash \mathbb{H} \times \mathbb{H}$ . *Internat. Math. Res. Notices*, (17):913–933, 2001.
- [Lin06] Elon Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)*, 163(1):165–219, 2006.
- [RS94] Zeév Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.
- [Sar] Peter Sarnak. Letter to morawetz,  
[http://www.math.princeton.edu/sarnak/sarnak\\_letter\\_to\\_morawetz.pdf](http://www.math.princeton.edu/sarnak/sarnak_letter_to_morawetz.pdf).
- [Sar11] Peter Sarnak. Recent progress on the quantum unique ergodicity conjecture. *Bull. Amer. Math. Soc. (N.S.)*, 48(2):211–228, 2011.
- [Zel87] Steven Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987.